Vector Integral Calculus

Terms of a Curve

Before I introduce the line integral, we need to get familiar with a few terms involving curves. Something new compared to scalar functions, vector functions that describe curves require a new set of definitions.

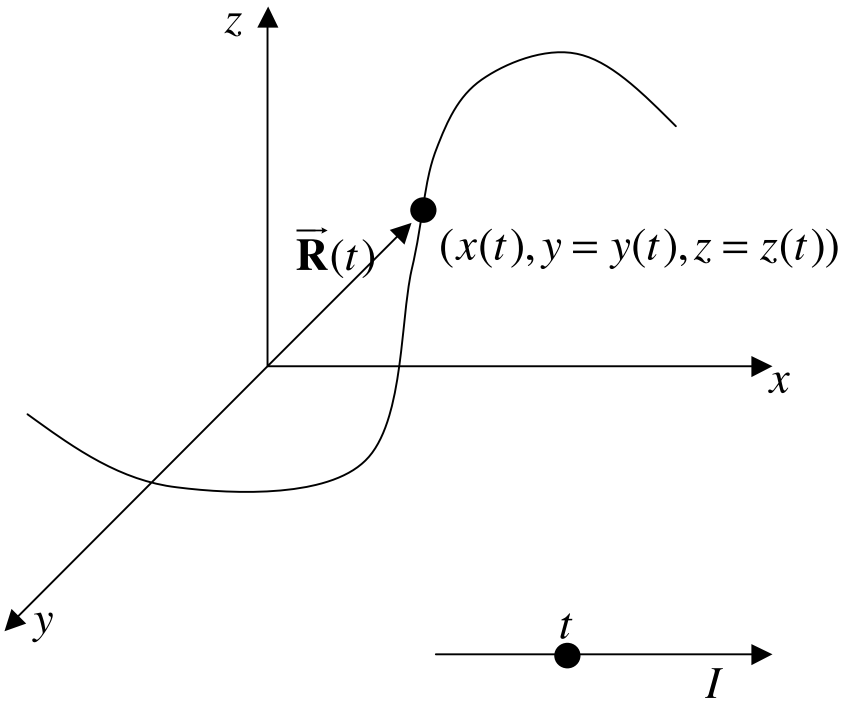
A curve in three-space, or three dimensional space, is usually given parametrically by the *coordinate functions*

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

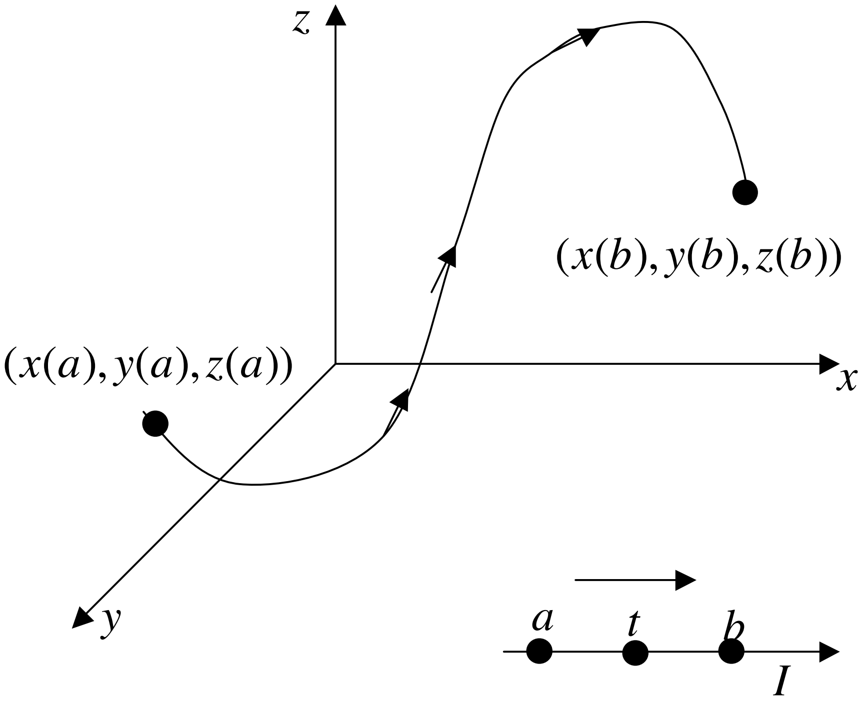
for some t in the interval *I*, which is the real line. We call t the parameter of the curve. To recap, the *graph* of a curve consists of the locus of points $(x(t), y = y(t), z = z(t))$ as t varies over *I*. We can write these coordinate functions in a *position vector* such as

$$\vec{\mathbf{R}} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \quad \text{for} \quad t \text{ in } I$$

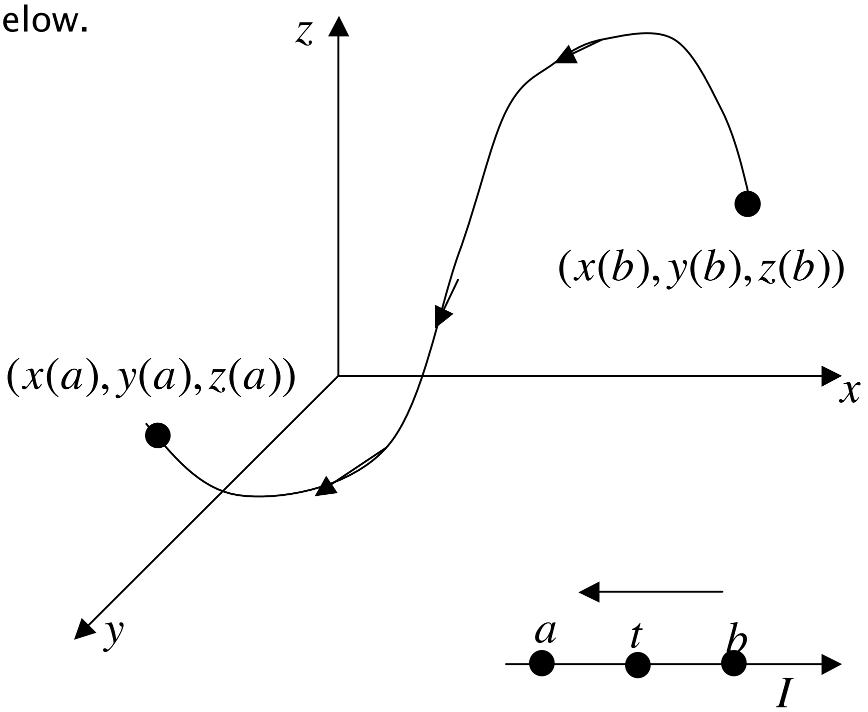
When we speak of the curve given by $\vec{\mathbf{R}}(t)$, we mean the curve whose coordinates are the components of $\vec{\mathbf{R}}(t)$. Here, $\vec{\mathbf{R}}(t)$ gives us the vectors from the origin to the points on the curve as t varies over *I* as shown below.



We now introduce the concept of the direction of which *t* increases and decreases. Suppose parameter *t* is defined over a closed interval $[a,b]$. We call the point $(x(a),y(a),z(a))$ the *initial point* of the curve and $(x(b),y(b),z(b))$ the *terminal point*. We get a sense of orientation along the curve as we move *t* from the initial to the terminal point, or as *t* varies from *a* to *b*. This orientation is indicated by an arrow on the graph.



We can also change the direction of *t* and let it vary from *b* to *a*, reversing the orientation of the curve moving from the new initial point $(x(b),y(b),z(b))$ to what we now call the terminal point $(x(a),y(a),z(a))$ as shown below.

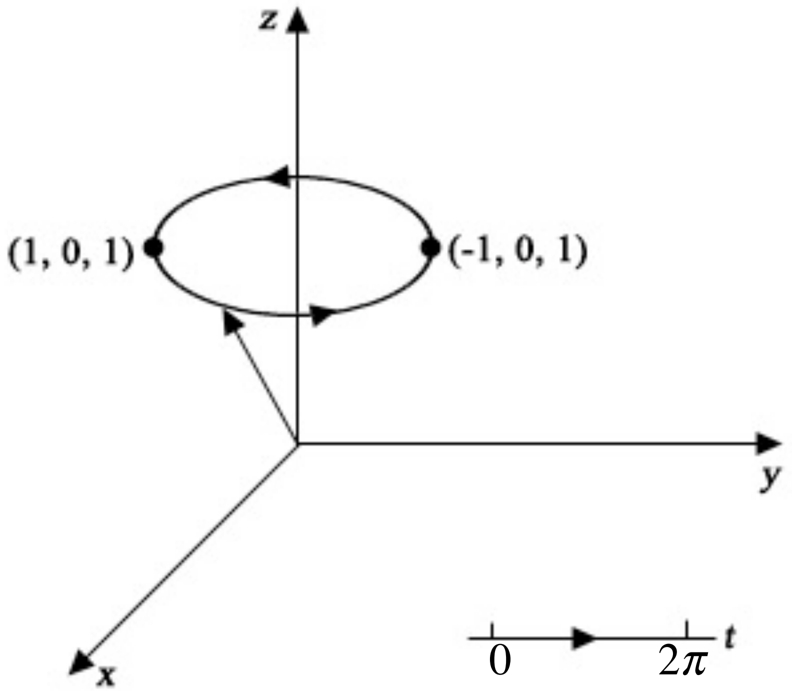


Unless indicated, we understand the orientation of a curve by letting t increase over its interval.

Let us look at a certain curve to have a firm understanding of the above terms. Let C be defined as

$$x = \cos(t), \quad y = \sin(t), \quad z = 1; \quad 0 \le t < 2\pi$$

From $x^2 + y^2 = 1$, C is a circle of radius 1 about the origin in the plane $z = 1$. As t increases from zero to 2π, the point $(x(t), y(t), z(t))$ moves around a circle through $(0,1,1)$ when $t = \pi/2$, through $(-1,0,1)$ when $t = \pi$, through $(0,-1,1)$ when $t = 3\pi/2$ and ending with $(1,0,1)$ when $t = 2\pi$. In this case, the initial and terminal points are the same. The graph is shown below.

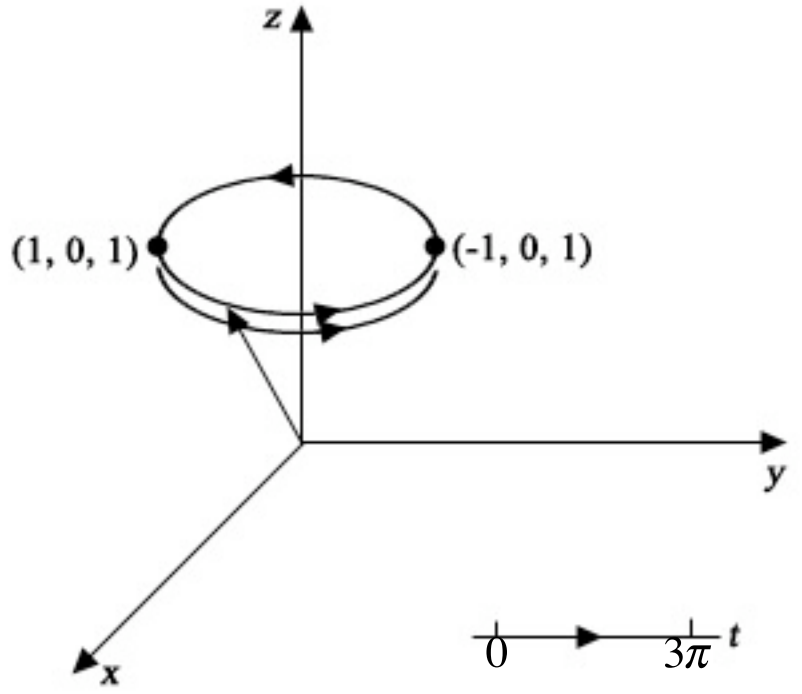


We call a curve C a *closed curve* if the initial and terminal points are the same.

Let's take it step further and change things up a bit. Another curve D is given by

$$x = \cos(t), \quad y = \sin(t), \quad z = 1; \quad 0 \le t < 3\pi$$

the only difference being that the interval in which *t* varies has increased to 3π. This time, when the point is at $(1,0,1)$ when $t = 2\pi$, it continues another semi circle to end at $(-1,0,1)$ when $t = 3\pi$. Although the graph takes the same appearance as curve *C*, it is *not* a closed curve because the initial and terminal points are not the same.



While this may seem primitive, it has vital importance when dealing with the line integral. Think about it in this way. The force, say a vector field, requires energy to push the point along the curve. It should be obvious that the energy requirement is DIFFERENT when pushing the point around one loop, in this case a closed curve, or one and a half loop. Thus it is important to know when a curve is closed.

It should be clear from the previous example that a curve is not the same as its graph. The curve comprises of three coordinate functions *and* the interval over which they are defined, along with a sense of direction on the curve. The graph is a locus of these points in three-space, not taking into account the interval.

If the coordinate functions are continuous on $[a,b]$, we call the curve *continuous*; and if they are differentiable, we call the curve *differentiable*.

Lastly, we wrap up with a few more terms (not like you have enough of them already). If $x'(t), y'(t), z'(t)$ are continuous on $[a,b]$ and are not all zero for any value of t, $\overrightarrow{\mathbf{R}}(t)$ is the tangent vector to the curve and is continuous as well. We then call this curve *smooth* which roughly speaking, does not have any sharp points in its graph.

Having defined all these new terms, we can now look at the line integral.