Vector Integral Calculus

Vector Fields and Lines of Force

We will begin our chapter of vector integral calculus with a short introduction of vectors fields and lines of force. In this lesson, we simply discuss some differences between vector calculus and normal calculus and briefly talk about some links the mathematics has with physics.

Up to now, we have stuck with vector functions of a single real variable, or a variable we call a parameter. In the topic of vector fields, we need to define a vector function of two variables. It takes the appearance of

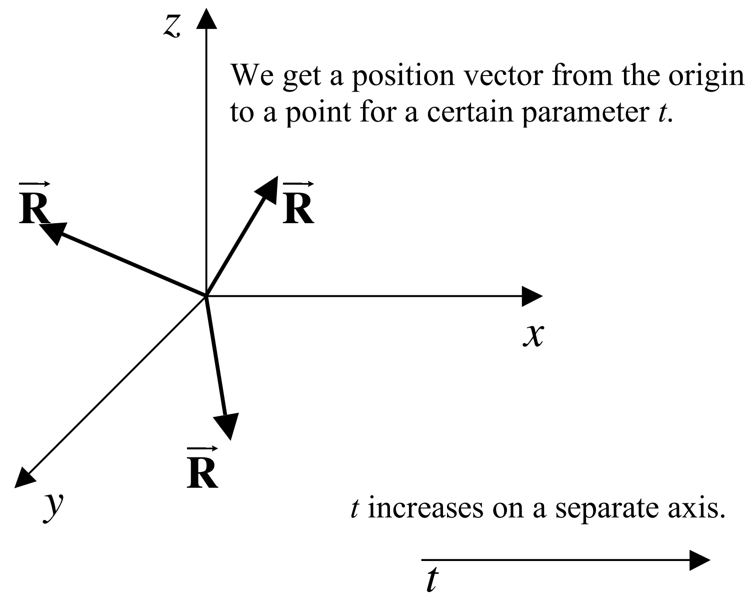
$$\vec{\mathbf{G}}(x,y) = g_1(x,y)\mathbf{i} + g_2(x,y)\mathbf{j}$$

for a plane where *x* and *y* are the two independent variables. In 3D space, a vector function has three variables and takes the form

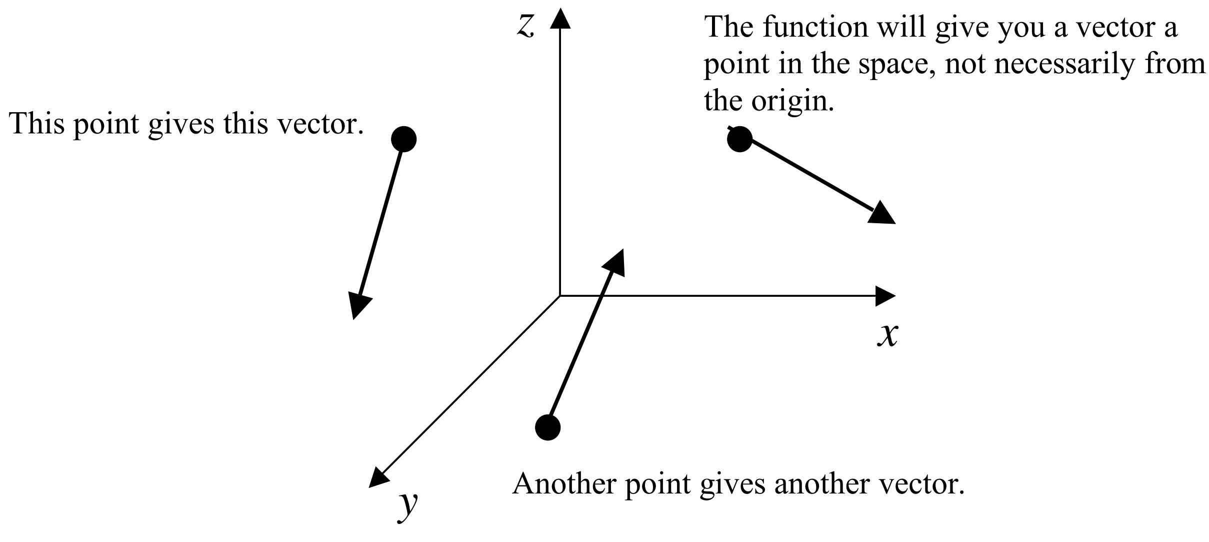
$$\vec{\mathbf{F}}(x,y,z) = f_1(x,y,z)\mathbf{i} + f_2(x,y,z)\mathbf{j} + f_3(x,y,z)\mathbf{k}$$

We call these functions *vector fields.* $\vec{\mathbf{G}}$ is a vector field in 2-dimensional space and $\vec{\mathbf{F}}$ is a vector field in 3-dimensional space. Unlike our previous functions of the position vector where we get a vector from the origin to a point in space for a given parameter, what the vector field means is that you give me a point *in* the space and I give you a vector. Notice the difference that the (x,y,z) corresponds to a point in the space and NOT a parameter on a separate axis, the t we have been using all along with our position vector.

Previously, we used a parameter *t* is define our position vector $\vec{\mathbf{R}}$ where $\vec{\mathbf{R}} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$.



Now for a vector field, you pick a point *in* the 3-dimensional space and the by the function $\vec{\mathbf{F}}(x,y,z) = f_1(x,y,z)\mathbf{i} + f_2(x,y,z)\mathbf{j} + f_3(x,y,z)\mathbf{k}$, it will give you a vector in that space.



So this begs the next question: *Is it possible to sketch a vector field?* The answer is no. Unlike a position vector where each vector is from the origin, a vector field generates a vector from *every* point in the space. This would give us an infinite amount of vectors which simply can't be sketch in the space.

A vector field is *continuous* if each of the component functions is continuous. We define the partial derivative of a vector field to be the vector field obtained by taking the partial derivative of each of the components. With $\vec{\mathbf{G}}(x,y)$ and $\vec{\mathbf{F}}(x,y,z)$ as defined previous, we have

$$\frac{\partial\vec{\mathbf{G}}}{\partial x}=\frac{\partial g_1}{\partial x}\mathbf{i}+\frac{\partial g_2}{\partial x}\mathbf{j} \text{ and } \frac{\partial\vec{\mathbf{G}}}{\partial y}=\frac{\partial g_1}{\partial y}\mathbf{i}+\frac{\partial g_2}{\partial y}\mathbf{j}$$

and for the 3-variable vector function,

$$\frac{\partial\vec{\mathbf{F}}}{\partial x}=\frac{\partial f_1}{\partial x}\mathbf{i}+\frac{\partial f_2}{\partial x}\mathbf{j}+\frac{\partial f_3}{\partial x}\mathbf{k}$$
$$\frac{\partial\vec{\mathbf{F}}}{\partial y}=\frac{\partial f_1}{\partial y}\mathbf{i}+\frac{\partial f_2}{\partial y}\mathbf{j}+\frac{\partial f_3}{\partial y}\mathbf{k}$$
$$\frac{\partial\vec{\mathbf{F}}}{\partial z}=\frac{\partial f_1}{\partial z}\mathbf{i}+\frac{\partial f_2}{\partial z}\mathbf{j}+\frac{\partial f_3}{\partial z}\mathbf{k}$$

You would find the full theory behind partial differentiation in a Calculus textbook. For now, we are just concern with the basic techniques and processes of differentiating and not the proof.

It should be obvious that a partial derivative of a vector field is again a vector field.

So how does this connect with physics? Well, the most common example is that of water flowing in a pipe. We imagine the water to be the 3D space we are concern with. The vector field $\vec{\mathbf{F}}(x,y,z)$ gives us the vector associated a point on a particular curve which a water particle is traveling on. Such curves are called streamlines, flow lines or more generally lines of force of the vector field. The term used varies with the context. For example, if $\vec{\mathbf{F}}(x,y,z)$ represents the velocity of a water particle, then the curve is designated as steams lines.

Our subsequent study is finding the equations of lines of force when we are given a certain vector field.