Multiple Integrals

Double Integrals over Nonrectangular Regions

Previously, we have restricted our double integral over a region R that is rectangular. This lessons looks at double integrals over nonrectangular regions.

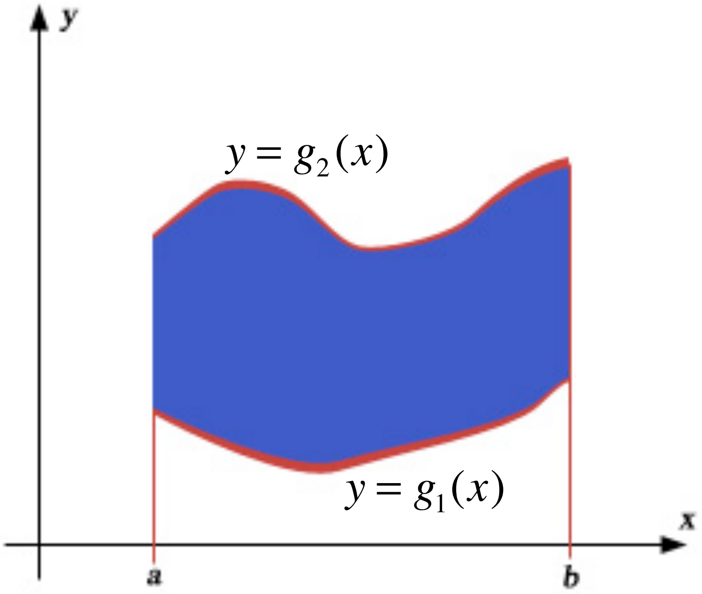
Double integrals over nonrectangular regions can often be reduced to iterated integrals of the following types:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx \quad \text{or} \quad \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

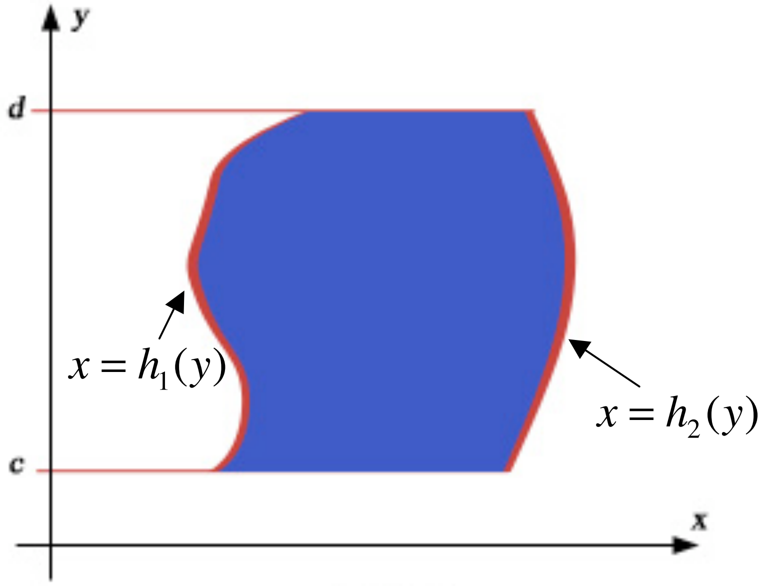
You may already notice that the limits of integration are now functions, $g_1(x), g_2(x), h_1(y), h_2(y)$. These functions describe the region, now that the region is nonrectangular. Since plane regions can be extremely complex, we shall limit our study to two basic types of regions which shall be called *type I* and *type II*.

A **type I region** is bounded on the left and right by vertical lines *x* = *a* and *x* = *b* and is bounded below and above by continuous curves $y = g_1(x)$ and $y = g_2(x)$, where $g_1(x) \le g_2(x)$ for $a \le x \le b$.

A **type II region** is bounded on the below and above by vertical lines *y* = *c* and *y* = *d* and is bounded left and right by continuous curves $x = h_1(y)$ and $x = h_2(y)$, where $h_1(y) \le h_2(y)$ for $c \le y \le d$.



TYPE I



TYPE II

The following theorem, somewhat similar to the fundamental theorem of calculus, enables us to evaluate double integrals over type I or type II regions using iterated integrals.

1. If R is a type I region on which *f*(*x*,*y*) is continuous, then

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

2. If R is a type II region on which $f(x,y)$ is continuous, then

$$\iint_R f(x,y)dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)dxdy$$

The usefulness of this theorem can't be underemphasized. If we need to find the double integral of a function $f(x,y)$, we can get the result use two repeated integrations provided that region R is described correctly depending on whether it is type I or type II.

Although we shall not formally prove this theorem, we can provide a geometrical argument if f(x,y) is nonnegative, in which case the double integral

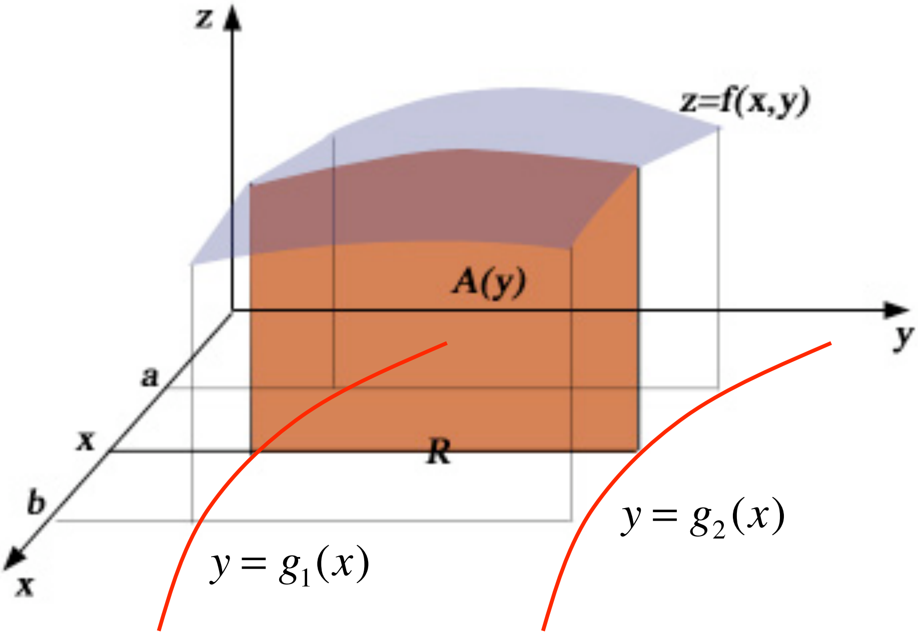
$$\iint_R f(x,y)dA$$

represents the volume of the solid *S* bounded above by the surface $z = f(x,y)$ and below by the region *R*. By the method of cross sections, the volume of *S* is also given by

$$\mathrm{Vol}(S) = \int_a^b A(x)dx$$

where $A(x)$ is the area of the cross section at the fixed point x. As shown below, this cross-sectional area extends from $g_1(x)$ to $g_2(x)$ in the y-direction. So

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x,y)dy$$



Substituting this into our volume equation, we obtain

$$\mathrm{Vol}(S) = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

And since this result is the volume of *S*, we obtain

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

A similar method can be used to prove the theorem for type II regions.

I do hope that by reading the steps of the argument above, one would understand why when we integrate the inner integrand wr.t to *y*, the limits needs to be in terms of *x*. You should see from the area function, *A*(*x*), it is a single integral along the *y*–axis and so we need to ‘describe’ the limits with functions *y* = *f*(*x*), just think instead of using *y* = *a* and *y* = *b*, we replace it with functions $y = g_1(x)$ and $y = g_2(x)$, due to the fact that is a nonrectangular region.

If you are still unclear about, don’t fret. The next two lessons, examples of evaluating the double integral over type I and type II regions, should explain spell out this discrepancy.