Multiple Integrals
Evaluating Double Integrals

In this lesson, we will show the process of obtaining a value for the double integral using two successive single integrations. We will limit our discussion to the case where *R* is a rectangle so as to get the 'feel' of the method before moving on to more complicated integrals.

Usually, it is impractical to obtain the value of the double integral by evaluating the limit, the right hand side of the equation,

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$

We need a more direct way to evaluate this double integral which will use a new technique of integration when dealing with two variable functions.

Recall that the partial derivatives of a function *f*(*x,y*) are calculated by holding one of the variables fixed and differentiating with respect to the other variable. Imagine that we will do this process, but in reverse, and call it *partial integration*. The symbol

$$\int_a^b f(x,y)\,dx$$

is a *partial definite integral with respect to x*. It is evaluated by holding *y* fixed and integrating with respect to *x*, treating *y* is a constant. Likewise, the *partial definite integral with respect to y*,

$$\int_c^d f(x,y)\,dy$$

is evaluated by holding *x* fixed and integrating with respect to *y*.

Let us see this partial integration in action on the same function *f*(*x,y*) but first with respect to *x* and then with respect to *y*.

$$\int_0^1 x^3 y dx = y\int_0^1 x^3 dx = y\left[\frac{1}{4}x^4\right]_0^1 = \frac{y}{4}$$

$$\int_0^1 x^3 y dy = x^3\int_0^1 y dy = x^3\left[\frac{1}{2}y^2\right]_0^1 = \frac{x^3}{2}$$

You may have already notice that we get different results depending of which term with we integrate with respect to. An integral of the form $\int_a^b f(x,y)dx$ produces a function of *y* as the result, while an integral of the form $\int_c^d f(x,y)dy$ produces one of *x*. Remember that in each case, the variable we are holding fixed is treated as a constant and so we can bring it outside the integral during evaluation.

We anticipate that our two single integrations will be done either as

$$\int_c^d\left[\int_a^b f(x,y)dx\right]dy$$

$$\int_a^b\left[\int_c^d f(x,y)dy\right]dx$$

For the first form, the inside integration, $\int_a^b f(x,y)dx$ leads to a function of *y*, which is then integrated with respect to *y*. For the second, integrating $\int_c^d f(x,y)dy$ leads to a function of *x* which is then integrated with respect to *x*. These expressions are called *iterated integrals* where the square brackets are usually removed. Below is an example.

We want to evaluate the expression,

$$\int_1^2\int_0^3 (2+9xy^2)dxdy$$

We integrate the function, in this case $f(x,y)=2+9xy^2$ first with respect to *x* and then with respect to *y*.

$$\int_1^2 \int_0^3 (2+9xy^2)dxdy = \int_1^2 \left[\int_0^3 (2+9xy^2)dx \right] dy$$

$$= \int_1^2 \left[2x + \frac{9}{2}x^2y^2 \right]_{x=0}^{x=3} dy$$

$$= \int_1^2 6 + \frac{81}{2}y^2 dy$$

$$= \left[6y + \frac{81}{6}y^3 \right]_1^2 = \frac{201}{2}$$

We round up by formally stating a theorem with links the double integral we initially wanted to find out with the two single partial integrations.

Let *R* be the rectangle defined by the inequalities $a \le x \le b,\ c \le y \le d$. If *f*(*x,y*) is continuous on this rectangle, then

$$\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$$

This theorem allows us to evaluate a double integral over a rectangle by calculating an iterated integral. Moreover, the theorem tells us that the order of the integration does not matter. Now, listen very carefully. This is only applicable over *rectangular regions* as specified by the limits *x* = *a* to *x* = *b* and *y* = *c* to *y* = *d*. As we shall later see, the order of integration is in fact important when dealing with nonrectangular regions, which comprises a large proportional of our subsequent lessons.

While, we shall not following proof the theorem, we can use a geometrical argument for the case where *f*(*x,y*) is nonnegative on *R*.

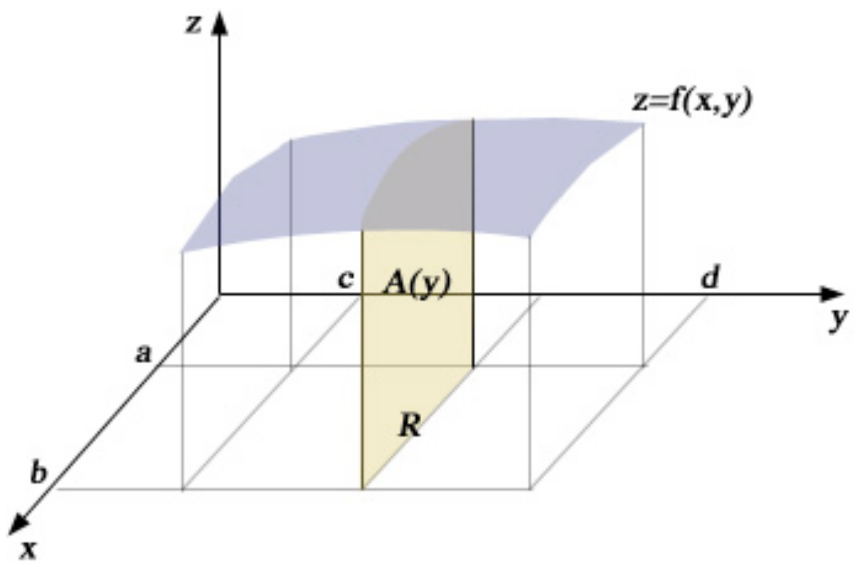
If *f*(*x,y*) is nonnegative on R, the double integral

$$\iint_R f(x,y)dA$$

represents the volume of the solid *S* bounded above by the surface $z=f(x,y)$ and below by the rectangle *R*. However, as one may notice, we may also write the volume of the solid *S* as

$$\text{Vol}(S) = \int_c^d A(y)dy$$

where $A(y)$ represents the area of the cross section of the perpendicular to the *y*-axis taken at the point *y*. So, how might we compute the cross-sectional area $A(y)$. This is the usual single variable integration problem. We fix *y* in the interval $c \le y \le d$ turning the function $f(x,y)$ into a function of *x* alone. $A(y)$ may be viewed as the area under the graph of this function along the interval $a \le x \le b$.



Thus,

$$A(y) = \int_a^b f(x,y)dx$$

Substituting this expression into our volume equation gives us

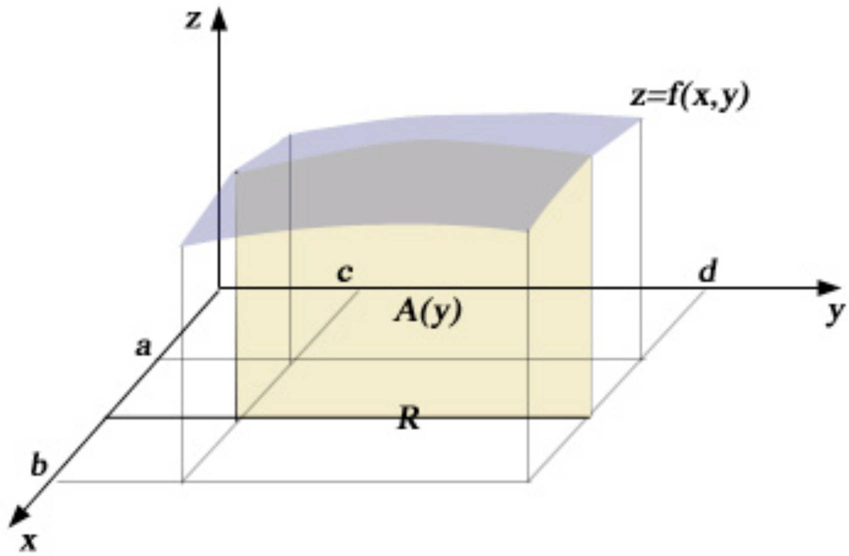
$$\text{Vol}(S) = \int_c^d \left[\int_a^b f(x,y)dx \right] dy = \int_c^d \int_a^b f(x,y)dxdy$$

We may also obtain the volume of solid *S* by using cross sections perpendicular to the *x*-axis. Using the similar formula,

$$\text{Vol}(S) = \int_a^b A(x)dx$$

where this time *A*(*x*) is the area of the cross section perpendicular to the *x*-axis taken at the point *x*. Using a similar process, we fix *x* in the interval $a \le x \le b$, the function *f*(*x*,*y*) is a now a function of *y* alone so that the area *A*(*x*) is given by

$$A(x) = \int_c^d f(x, y)dy$$



Substituting this expression into our volume equations gives us

$$\text{Vol}(S) = \int_a^b \left[\int_c^d f(x, y)dy \right] dx = \int_a^b \int_c^d f(x, y)dydx$$

Since the volume of *S* is also given by the double integral $\iint_R f(x, y)dA$, it follows from our previous results that

$$\iint_R f(x, y)dA = \int_c^d \int_a^b f(x, y)dxdy = \int_a^b \int_c^d f(x, y)dydx$$

which is what we initially set out to show.