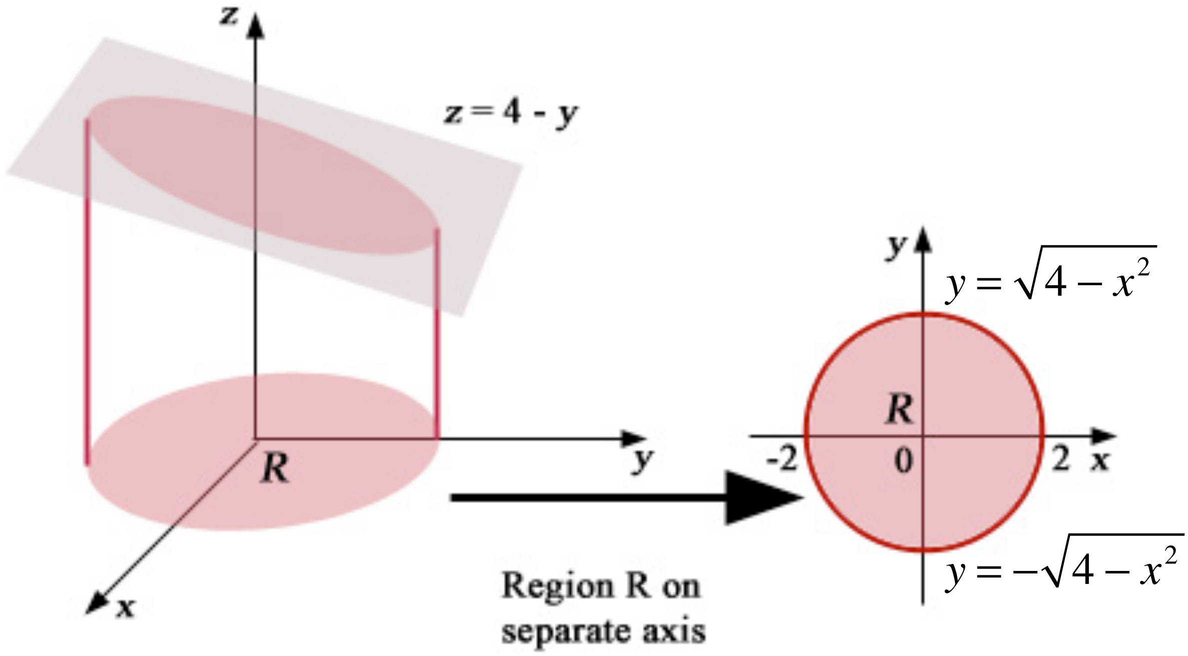
Multiple Integrals
Finding Volume Problems (Cylinder)

We now move to a slightly more difficult lesson where we use the double integral as a technique to find the volume of solid bounded by a cylinder and a plane.

Our task is to find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

First, we should get a rough idea of the solid we are dealing with. One may immediately notice the equation of the *cylinder* is given by $x^2 + y^2 = 4$, an equation we all use to describe a circle. Yes, it is an equation of a circle, but in $\mathbb{R}^3$ space, we basically get all the circles at the different values of *z*. And when you join them together, you get a cylinder. The lack of the dependence on *z* tells us that we take the circle in the *xy*–plane and just extend it infinitely up and down, parallel to the *z*–axis.

So we get our solid by intersecting this cylinder with the given planes $z = 4 - y$ and $z = 0$, as shown below.



As always, the volume of the solid is given by

$$V = \iint_R (4 - y)\,dA$$

Now comes the somewhat tricky task of describing the region *R* using the correct *x* and *y* limits. The equation of the circle enclosing *R* is $x^2 + y^2 = 4$. We need to write the limits for *y* as a function in terms of *x*. Initially, one may be quick to rearrange the equation to give use the function $y = \sqrt{4 - x^2}$ that describes the top of the circle. But what about the other limit? It would be a mistake to think of it as $y = 0$. Instead, it must be the function in terms of *x* that describes the lower part of the circle. We get this by taking the negative square root, $y = -\sqrt{4 - x^2}$. The *x*-limits are pretty straight forward.

Now everything is in place.

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - y)\,dy\,dx = \int_{-2}^{2} \left[4y - \frac{1}{2}y^2 \right]_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} 8\sqrt{4 - x^2}\,dx = 8(2\pi) = 16\pi$$

The volume of our cylinder enclosed by the given planes is 16π units cube.