Multiple Integrals

**Finding Volume Problems (Tetrahedron)**

This lesson and the next are about using the double integral as a technique to find the volume of certain solids.

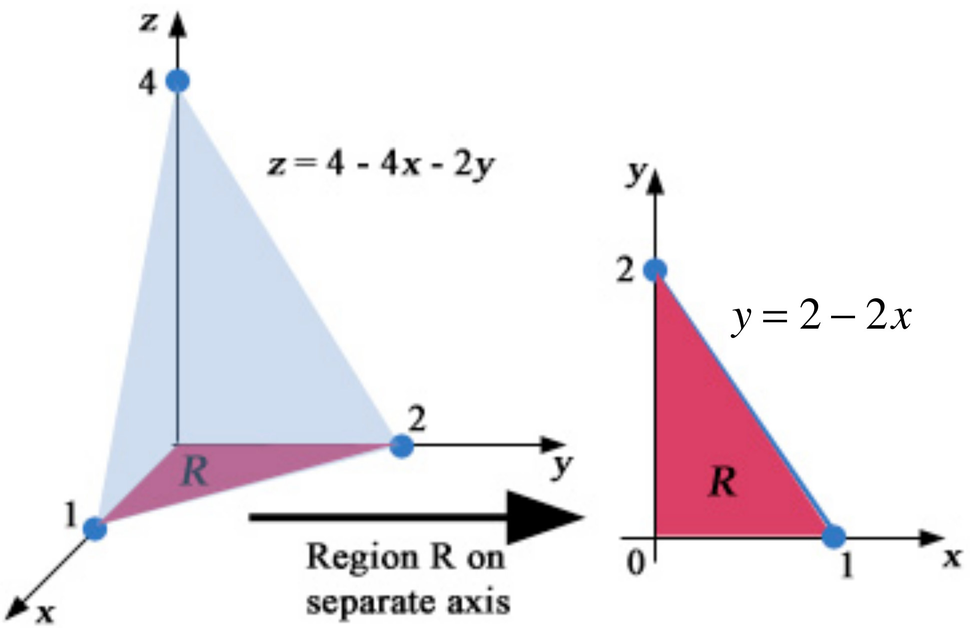
We start of by using the double integral to find the volume of the tetrahedron bounded by the coordinate planes and the plane $z = 4 - 4x - 2y$.

While it is as though we are started out with very little information regarding our solid, it is in fact sufficient to define it.

The tetrahedron in question is bounded above by the plane,

$$z = 4 - 4x - 2y$$

and below by the triangular region *R* as shown below. If you are having trouble picturing the region *R*, you can think of it as this way. Obviously, the equation of the plane, $z = 4 - 4x - 2y$, extends to infinity in the $\mathbb{R}^3$ space. We get our solid by enclosing the space between the plane, *xy*–plane, *yz*–plane and the *zx*–plane, what we call a tetrahedron. The region *R* is the projection of the solid onto the *xy*–plane and the equation of intersection is simply gotten by letting $z = 0$.




Remember that when dealing with double integrals, we have the function, in this case the function of the plane $z = f(x, y)$, and the region *R*. So with *R* defined as such, we can now calculate the volume.

$$V = \iint_R (4 - 4x - 2y)\,dA$$

The region $R$ is bounded by the $x$–axis, the $y$–axis, and the line $y = 2 - 2x$ (again by setting $z$ = 0). So in treating $R$ as a type I region gives us

$$V = \iint_R (4 - 4x - 2y)\,dA = \int_0^1 \int_0^{2-2x} (4 - 4x - 2y)\,dy\,dx$$

$$= \int_0^1 \left[4y - 4xy - y^2\right]_{y=0}^{2-2x} dx = \int_0^1 \left(4 - 8x + 4x^2\right) dx \quad 2$$

$$= \frac{4}{3}$$

And so the volume of our tetrahedron is $4/3$ unit cube. Student may have difficult properly defining the region $R$. Thus it is at times useful to sketch the region $R$ on the $xy$–plane as I did above.