Multiple Integrals

Hunting Gold, Alternative Approach

Even though there weren't any gold at our targeted location, I still decided to look at the mathematics of the problem and explored other possibilities of solving this double integral problem.

In our initially method of solving this problem, we decided to break the region *R* into three sub-regions. Our final answer resulted in calculating the double integral of the function $z=f(x,y)$ for each of the sub-regions and then summing them, that is

$$\iint_R xy^2 dA = \iint_{R_1} xy^2 dA + \iint_{R_2} xy^2 dA + \iint_{R_3} xy^2 dA$$

While this method indeed works, we ended up with some complicating numerical values for our answer. Let's see whether there is a less tedious solution in terms of evaluating the integrals.

A quick look at the region *R* and we see that we can divide this region in a different way. We know that the double integral gives us a signed volume. So just like how we summed the individual sub-regions, why not we try subtracting sub-regions from say a bigger sub-region which ultimately gives us back the region *R*.

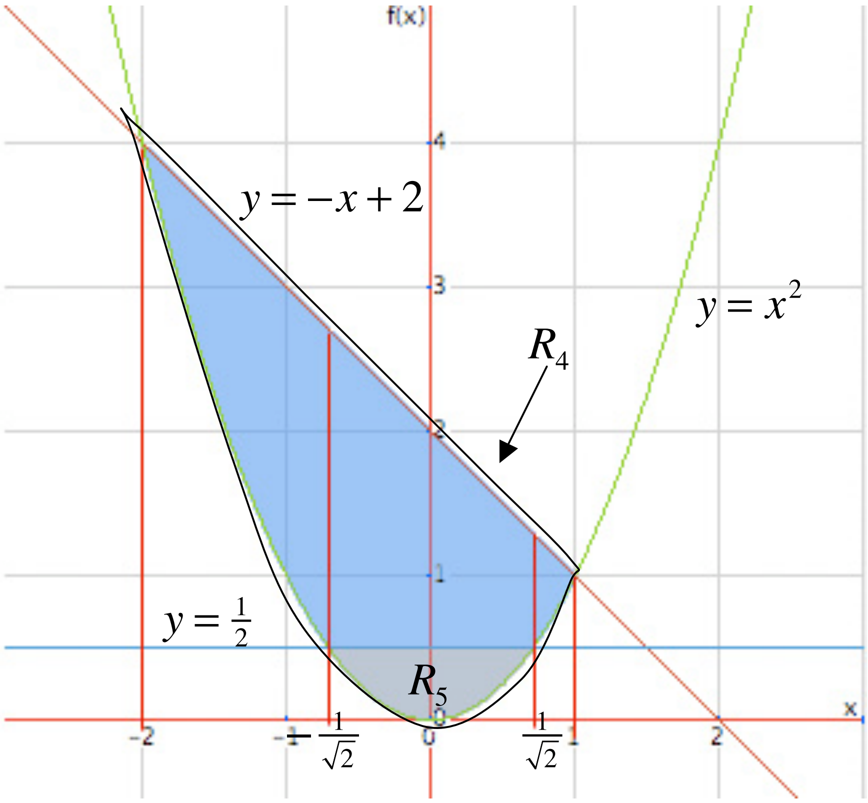
From the diagram, we can say

$$R = R_4 - R_5$$

where R_4 is the sub-region bounded by $y=-x+2$ and $y=x^2$ and R_5 is that bounded by $y=x^2$ and $y=\frac{1}{2}$. So the double integral of region *R* can now be expressed as

$$\iint_R xy^2 dA = \iint_{R_4} xy^2 dA - \iint_{R_5} xy^2 dA$$

Is this fine? Yes it is simply because we are adding or subtracting signed volumes accordingly to get our desired region. Moreover, we are dealing with just two double integrals now, one less from before.



Moreover, notice that both sub-regions are bound by exactly two $y=f(x)$ equations. There is no issue of intersecting points in these regions. So proceeding with the calculations,

$$\iint_{R_4} xy^2 dA = \int_{-2}^{1}\int_{x^2}^{-x+2} xy^2 dydx = \int_{-2}^{1}\left[\frac{1}{3}xy^3\right]_{x^2}^{-x+2} dx$$

$$= \frac{1}{3}\int_{-2}^{1} -x^7 - x^4 + 6x^3 - 12x^2 + 8x dx$$

$$= -\frac{603}{40}$$

Hmmm, that’s a little peculiar. We already got our answer without subtracting the answer from R_5. We may be quick to find an error in the calculation, but there’s isn’t. Evaluating the double integral for R_5,

$$\iint_{R_5} xy^2 dA = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{x^2}^{1/2} xy^2 dy dx = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left[\frac{1}{3} xy^3 \right]_{x^2}^{1/2} dx$$

$$= \frac{1}{3} \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{x(1-8x^6)}{8} dx$$

$$= 0$$

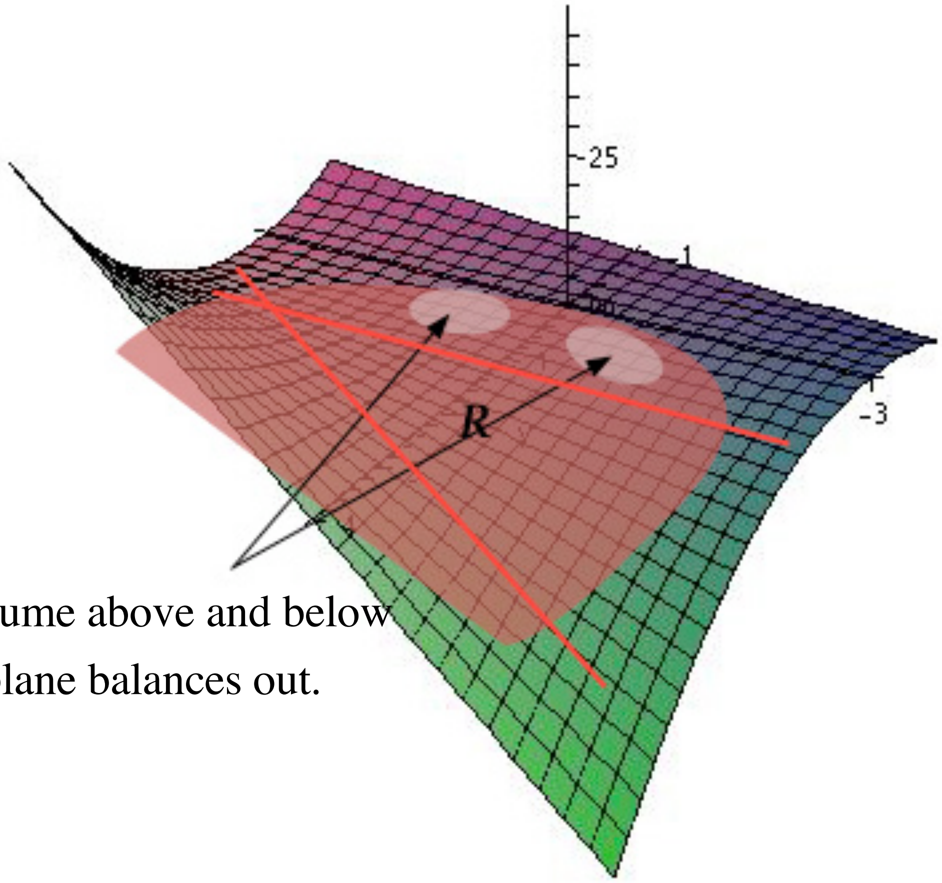
and thus

$$\iint_{R} xy^2 dA = \iint_{R_4} xy^2 dA - \iint_{R_5} xy^2 dA = -\frac{603}{40} - 0 = -\frac{603}{40},$$

which is the same result as before. So we may now wonder why the double integral evaluated over the region R_5 is 0. Well, there is an easy explanation.

The double integral gives us a signed volume. Now obviously, we know that in the sub-region R_5 as bounded by $y = x^2$ and $y = \frac{1}{2}$, the function (height function in this case) cannot be $z = 0$, but when given as $z = xy^2$, we either get a positive or negative value for z in this region. So, the positive signed volume and negative signed volume in R_5 will perfectly balance out when we combine them together resulted in 0 volume.

Be mindful that this does not mean no volume. It means that the volume above and below the *xy*-plane are equal. We can also say that the function $z = xy^2$ will cut the plane in this region $z = xy^2$, as graphically shown below.



The volume above and below the *xy* plane balances out.

So there you have it. An alternative way for our hunting for gold problem. I emphasize again that the negative volume suggest to us a cave in the designated location and so cranes are needed to dig for the gold.

Then again, you could just focus on the mathematics and leave the gold digging to others.