Multiple Integrals
Interpretations and uses of the Double Integral

With already six lessons on double integrals under our belt, we shall now have a short lesson on the interpretations and uses of the double integral, just to make sure we know what the concept is about.

Let us first contrast the double integral with the single integral.

The double integral is written as

$$\iint_R f(x,y)dA,$$

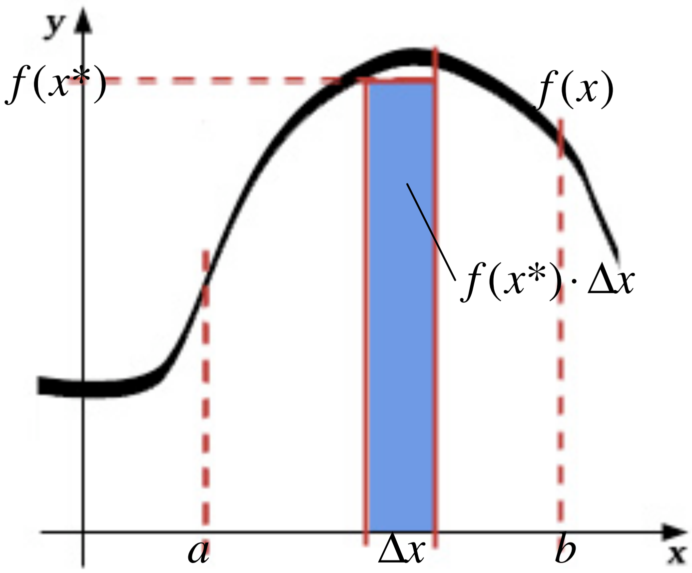
which is different with our all so familiar single integral

$$\int_a^b f(x)dx.$$

Now, first and foremost, we immediately notice the difference in the functions involved in the integration specifically the amount of independent variables each one uses. For the double integral, there are two independent variables – *x* and *y*. For the single integral, there is only one – *x*.

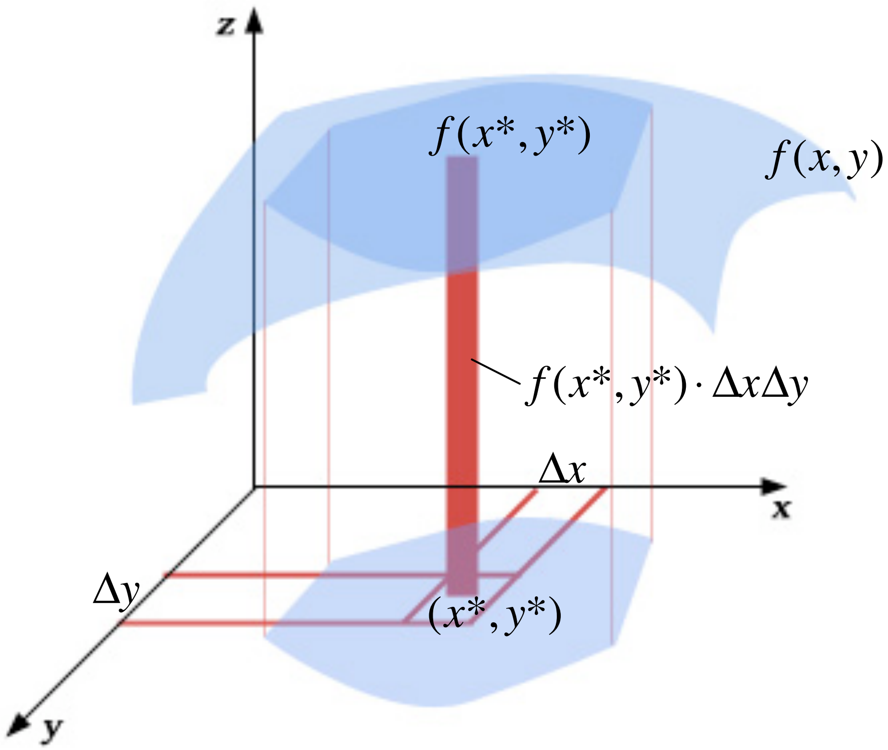
Next, we see that for the double integral, we are always integrating over the region *R* and for the single integral, from *a* to *b*. There is no such thing of a double integral where we are integrating $f(x,y)$ from *a* to *b or* from *c* to *d*. It is always over the region *R*.

We may also gain insights of each integral by looking at them geometrically.



For the single integral, as shown above, we are dealing with lines or curves as defined by the function $y = f(x)$. We should also know by now that the single integral is a limiting process where we divide the interval $[a,b]$ into *n* strips of width Δx and sum the quantity $f(x)\cdot\Delta x$ for each value of *x* in the interval $[a,b]$ and then let $n \to \infty$. This ultimately gives us the area under the curve.

Now, what about the double integral as shown below.



Geometrically for the double integral, we are summing the volume of small parallelepipeds whose height is given by the function $y = f(x,y)$ and the area is given by ΔA. Summing over the whole region *R*, and we ultimately get the volume bounded between the surface and the region *R*.

Moving back to the single integral for a moment, most of us would be somewhat familiar of its applications to physical problems. We should know the all too familiar phrase of 'take a small quantity and then integrate'. Supposing the amount of atoms in system a given energy is

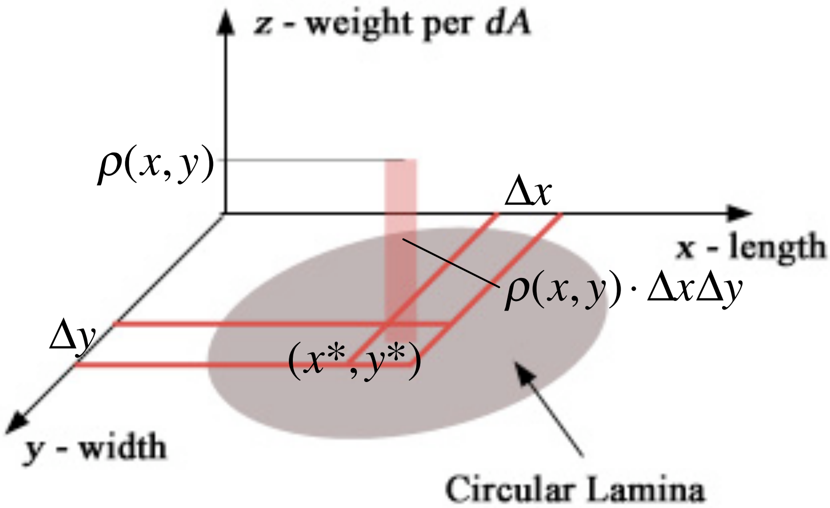
$$\frac{2\pi V}{h^3}\varepsilon^{1/2}d\varepsilon$$

Then, if we want to find the total amount of atoms, *N*, in a system covering all energy intervals from 0 to *E*, we perform the integration,

$$N = \int_0^E \frac{2\pi V}{h^3} \varepsilon^{1/2} d\varepsilon$$

The double integral has a physical interpretation that parallels this idea but this time, we are at liberty to vary two variables. Now, we can specify a function, or quantity, that changes accordingly based on two independent variables, unlike the single integral where we could only vary one variable.

Some of the most common physical uses are finding volumes, moment of inertia, and the weight of a laminar. What defines the physical uses is based on the information the function gives use. Previously, the function $y = f(x, y)$ gave us the height and so when we integrate over the region *R*, we get the volume. But now, if we use the function $y = \rho(x, y)$, where the density $\rho(x, y)$ gives us the weight at each point, and integrate over the region *R*, we get the weight of the whole laminar.



We may feel joyous knowing that we have this rather useful integration technique under our hands. Yes, it is definitely useful both in mathematics and physics. But I still like to caution the user to know what the function $y = f(x, y)$ means before doing the integration because getting this fact wrong would lead to wrong physical interpretations.

Especially for an applied mathematician, when the results of a calculation needs to be interpreted correctly!