Multiple Integrals

Limits of Region R (Type I)

Last lesson ended with a theorem on evaluating double integrals over a nonrectangular region *R*. In this lesson we will go through the steps in finding the limits for a type I region, applying the theorem and obtaining a value for the double integral.

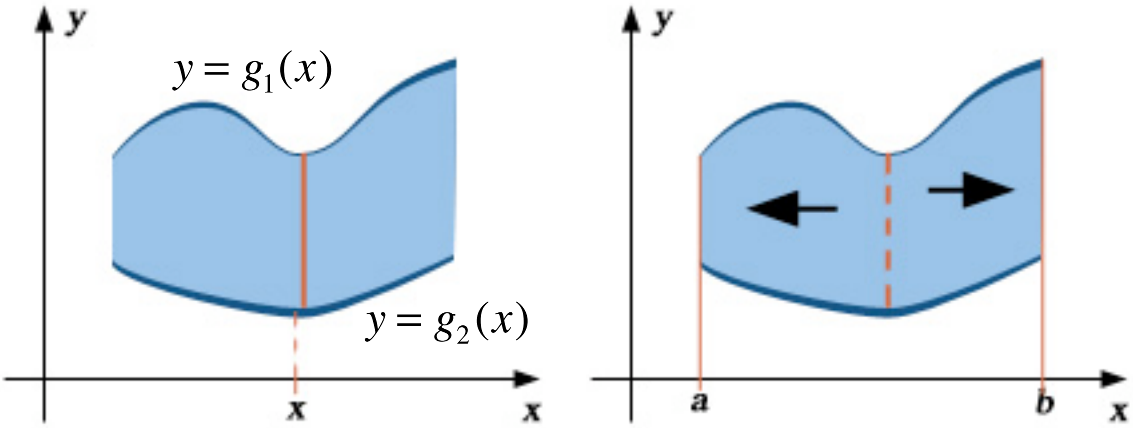
Since we are rather new at double integrals, it is highly advisable that we make a two-dimensional sketch of region *R*. The function *f*(*x*, *y*) need be sketch as it does not play a role in defining the limits*.

For a type I region, we'll use a simple two-step process.

Step 1:
We draw a vertical line through region *R* at an arbitrary fixed point *x*. This line crosses the boundary of *R* twice (see below). The lower point of intersection is on the curve $y = g_1(x)$ and this is the lower *y*-limit. The higher point of intersection is on the curve $y = g_2(x)$ and this is the upper *y*-limit.

Step 2:
Imagine moving the vertical line drawn in step 1 to left and then to the right. The leftmost position where the line intersects the region *R* is $x = a$ and this is the lower *x*-limit. The rightmost position where the line intersects the region *R* is $x = b$ and this is the upper *x*-limit.

As one gets familiar with defining limits, these steps need not be strictly followed and can be replaced with usual inspection.



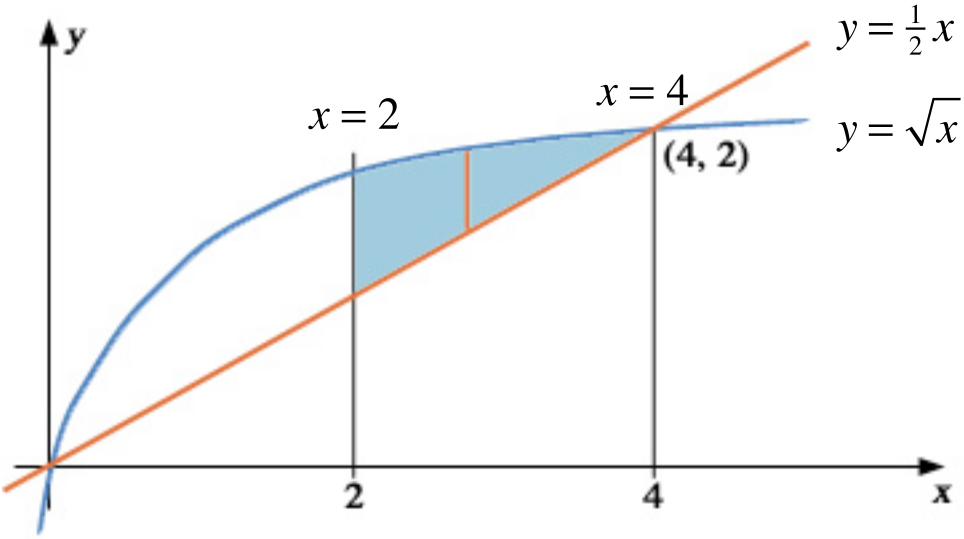
We now put these steps into practice by an example.

Let us evaluate

$$\iint_R xy\,dA$$

over the region *R* enclosed between $y=\frac{1}{2}x$, $y=\sqrt{x}$, $x=2$ and $x=4$.

We view *R* as a type I region. Region *R* and a vertical line corresponding to a fixed *x* is shown below.



This line meets the region *R* at the lower boundary $y=\frac{1}{2}x$ and the upper boundary $y=\sqrt{x}$. These become our *y*–limits of integration. Moving this line horizontally gives us our *x*–limits of integration, $x=2$ and $x=4$. Thus, changing into the iterated integral,

$$\iint_R xy\,dA=\int_2^4\int_{x/2}^{\sqrt{x}} xy\,dy\,dx=\int_2^4\left[\frac{xy^2}{2}\right]_{y=x/2}^{\sqrt{x}}dx$$

$$=\int_2^4\left(\frac{x^2}{2}-\frac{x^3}{8}\right)dx$$

To prevent any ambiguity, notice that the bottom limit to be evaluated at the square brackets is written as $y=x/2$. This is to tell us that we are substituting this limit '$x/2$' into where *y* is as this corresponds to the *y*–limits integration.

It should be clear now why the y-limits are written in terms of x. After substituting the y-limits we have eliminated y from the integrand ending up with an expression solely in terms of x so that the last step – integrating w.r.t to x – can be carried out.

Proceeding with the calculations,

$$\int_2^4 \left(\frac{x^2}{2} - \frac{x^3}{8}\right)dx = \left[\frac{x^3}{6} - \frac{x^4}{32}\right]_2^4 = \left(\frac{64}{6} - \frac{256}{32}\right) - \left(\frac{8}{6} - \frac{16}{32}\right)$$

$$\frac{11}{6}$$

It worth noting that when finding double integrals over nonrectangular regions, the iterated integral

$$\int_{x/2}^{\sqrt{x}} \int_2^4 xydxdy \neq \int_2^4 \int_{x/2}^{\sqrt{x}} xydydx$$

Unlike rectangular regions, the order of integration IS important. A little later, we'll see how we can reverse this order. For now, let's get aquatinted with process of specifying limits.

*We shall see in the later lesson 'Hunting Gold, a double integral problem', that $f(x, y)$ may affect how we define region R depending on the context and meaning of $f(x, y)$, particularly in the case when $f(x, y)$ crosses the x–y plane of region R. However, for most freshman courses in multivariable calculus, such difficult cases don't arise.