Multiple Integrals
Limits of Regions R (Type II)

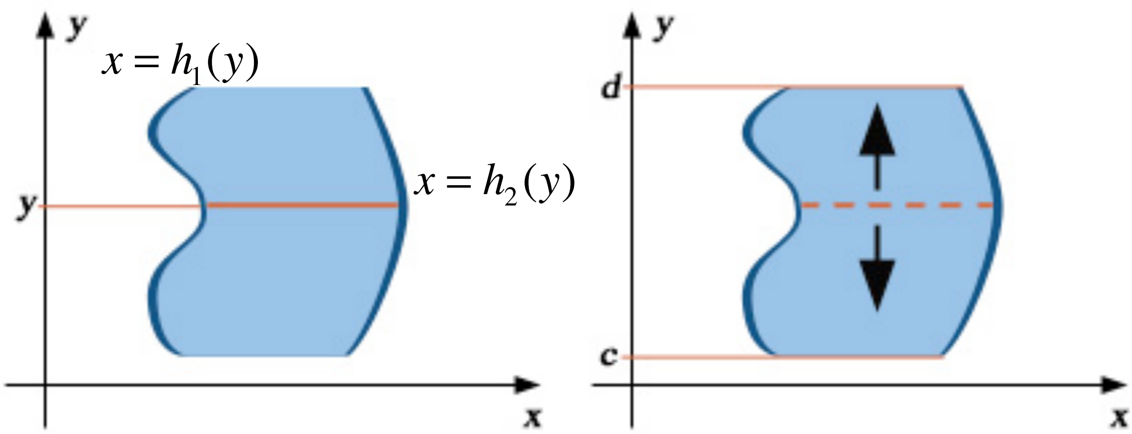
Similar to the previous one, this lesson will cover the steps in finding the limits for a type II region, leading to a result for the double integral.

Type II regions are different in that they describe *R* using the curve $x = x(y)$ and constant *y* limits. As a result, the region is usually curved at the left and right and not top and bottom. To find the limits, we'll use a similar two-step process.

Step 1:
With *y* held fixed for the first integration, draw a horizontal line through the region *R* at a fixed-point *y*. This line will cross the boundary of *R* twice. The leftmost point of intersection is on the curve $x = h_1(y)$, which is the lower *x*-limit of integration. The right point of intersection is on the curve $x = h_2(y)$, which is the upper *x*-limit.

Step 2:
Imagine moving the vertical line drawn in step 1 down and up. The lowest position where the line intersects the region *R* is $y = c$ and this is the lower *y*-limit. The highest position where the line intersects the region *R* is $y = d$ and this is the upper *y*-limit.

As one gets familiar with defining limits, these steps need not be strictly followed and can be replaced with inspection.



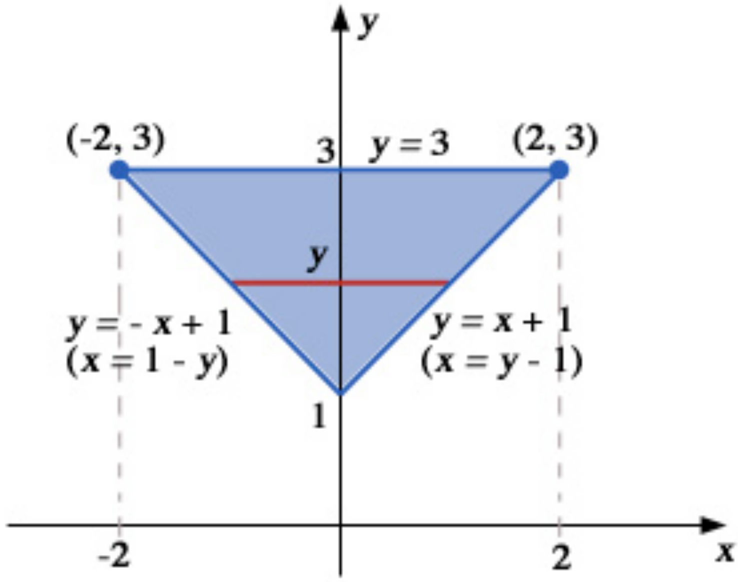
We now put these steps into practice by an example.

Let us evaluate the double integral

$$\iint_R (2x - y^2)\,dA$$

over the triangular region *R* enclosed between the lines $y = -x + 1$, $y = x + 1$ and $y = 3$.

We view *R* as a type II region. At this stage, it is not essential that we sketch out the function *f*(*x*,*y*). We may not know what this surface looks like but the surface does not affect the limits of region *R*, as given by the question. One could graph out region *R*, as shown below, to see the steps in work.



The red line is our horizontal line corresponding to a fixed *y*. This line meets the region *R* at its left boundary $x = 1 - y$ and the right boundary $x = y - 1$. We get these equations by rearranging the given equations to have *x* expressed in terms of *y*. Remember, type II regions are described with $x = h(y)$. Moving this line vertically down and up gives our lower and upper *y*-limits, $y = 1$ and $y = 3$. So,

$$\iint_R (2x-y^2)dA = \int_1^3 \int_{1-y}^{y-1} (2x-y^2)dxdy = \int_1^3 \left[x^2 - y^2 x\right]_{x=1-y}^{1} dy$$

$$= \int_1^3 \left[(1-2y+2y^2-y^3)-(1-2y+y^3)\right]dy$$

$$= \int_1^3 (2y^2-2y^3)dy$$

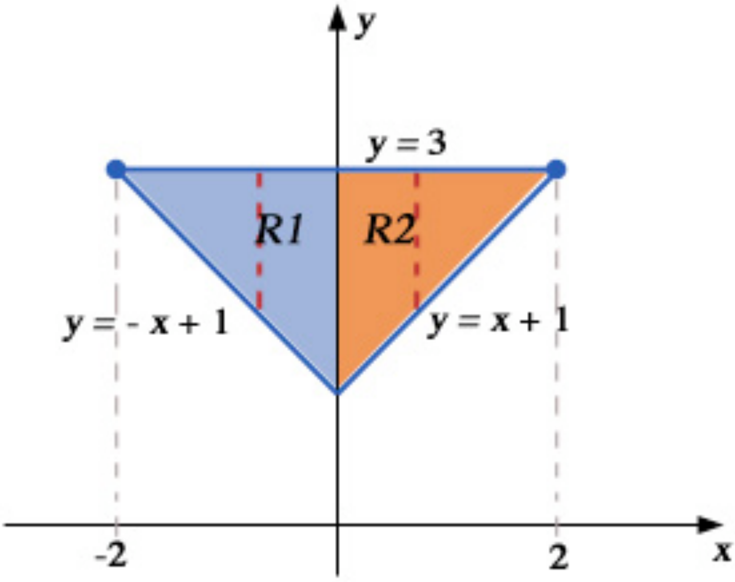
Remember, for our first integration, we are partially integrating with respect to *x*, treating *y* as a constant. Care is taken to write the limits for the evaluated integrand as *x* = 1 – *y*, because we are substituting the limits into where *x* is and not *y*. By doing so, we get an expression solely in terms *y* so that we can perform the final integration. We see yet again why when we integrate with respect to a certain term, in this case *x*, the limits are written in terms of *y*.

Finishing up with the calculations,

$$\int_1^3 (2y^2-2y^3)dy = \left[\frac{2y^3}{3} - \frac{y^4}{2}\right]_1^3 = -\frac{68}{3}$$

To reemphasize again, boundaries for type II regions must expressed in the form $x = h_1(y)$ and $x = h_2(y)$. Equations of graphs are usually given to us in the form $y = y(x)$ meaning that rearranging of equations to write them in terms of *x* are needed.

In this same example, we could have treated *R* as a type I region, with a small added complication. Viewed as a type I region, the upper limit of *R* is the line $y = 3$ and the lower limit consist of two parts, the line $y = -x+1$ to the left of the origin and the line $y = x+1$ to the right of the origin. We need to decompose the region *R* into two parts *R1* and *R2*, as shown below.



Our double integral is now

$$\iint_R (2x - y^2)dA = \iint_{R1} (2x - y^2)dA + \iint_{R2} (2x - y^2)dA$$

$$= \int_{-2}^{0}\int_{-x+1}^{3} (2x - y^2)dydx + \int_{0}^{2}\int_{-x+1}^{3} (2x - y^2)dydx$$

Continuing with the calculations (which I'll leave for the enthusiast reader), we will get the same result. Just please note that we are now *first* integration with respect to *y* and so our limits are written in terms of *x*, consistent with the steps of the previous lesson and the graph.