

Multiple Integrals

Reversing the order of integration

When evaluating double integrals, we can either integrate with respect to y first and then x or vice versa. In this short lesson, we will learn the method of reversing the order of integration.

Suppose we have the situation of finding the double integral of a type 2 region that is,

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y) \, dx dy$$

Clearly, this double integral wants us to integrate w.r.t x first and then y . Sometimes, this is a straight forward process. However, in other times, the function $f(x,y)$ may not be readily integrated w.r.t x but is easily integrated w.r.t to y . So, we must change the order of integration to get through the problem.

Given the multitude of functions, both the function of $f(x,y)$ and the x limits, we may be presented with many cases. Still, here are three basic steps you can follow.

1. Leave $f(x,y)$ alone.
2. use $x = h_1(y)$, $x = h_2(y)$ to express the region in terms of the other variable.
3. Graph R to rewrite limits (may not be necessarily if experienced enough)

As this is not a change of coordinate system, do not change the function $f(x,y)$. We are still sticking with the same coordinate system used to describe the function, just changing the order used to carry out the calculation.

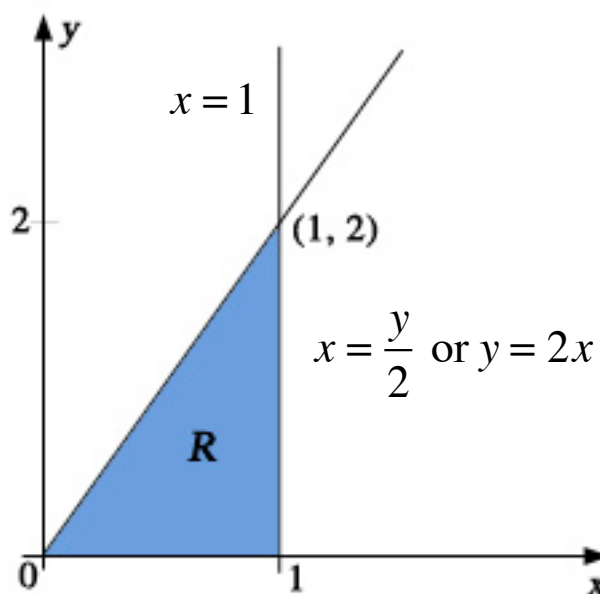
Instead, what changes is how we express the function as now we will write the same function in terms of the other variable. If we are reversing the order of integration to a type 1 region, that is integrated with respect to y first and then x , we would need the equations $y = g_1(x)$ and $y = g_2(x)$ which we get from a simple arrangement of $x = h_1(y)$ and $x = h_2(y)$.

Lastly, I advise that one graphs out the region in order to identify the new limits. After experience, this step can be either done with simple algebraic manipulations or with inspection.

For an example, let us evaluate,

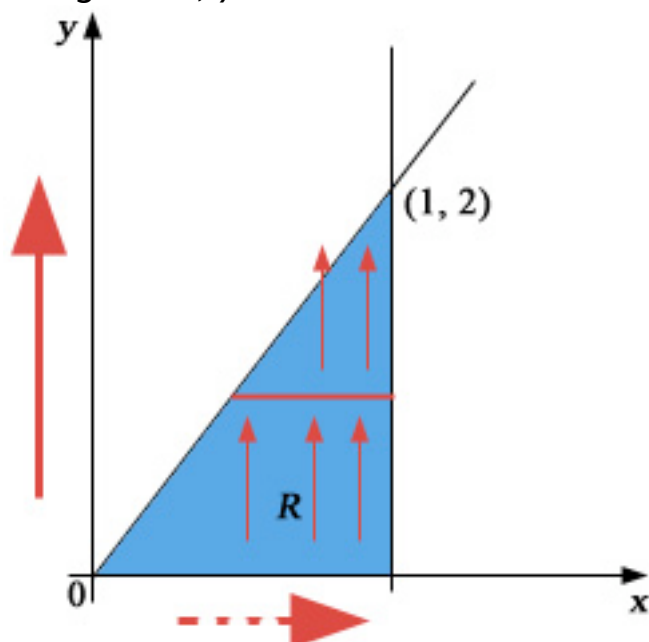
$$\int_0^2 \int_{y/2}^1 e^{x^2} dx dy$$

A sketch of the region R is below.



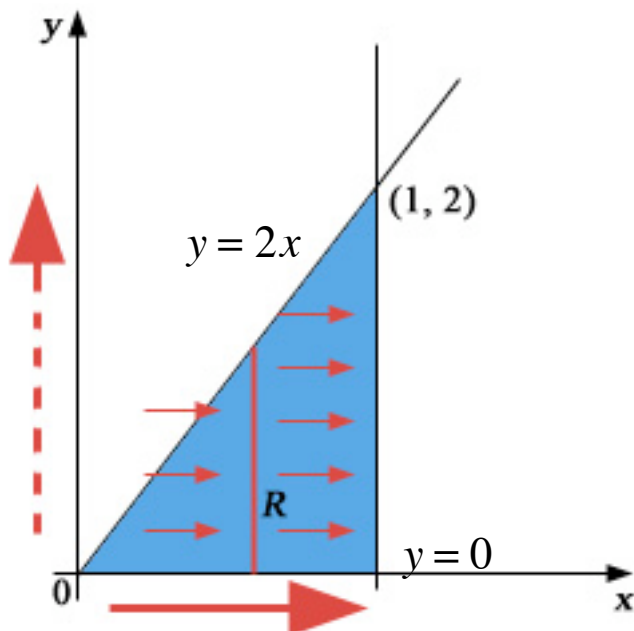
Since there is no elementary antiderivative of e^{x^2} , the integral cannot be evaluated by performing the x -integration first. What we do is to evaluate this integral by expressing it as an equivalent iterated integral with the order of integration reversed.

For the inside integration, y is fixed x varies from the line $x = y/2$ to the line $x = 1$. For the outside integration, y varies from 0 to 2.



This is a type 2 region or how we would describe the region is by drawing a horizontal line, tracing the left and right y -limits first and then moving the horizontal line up and down to trace the x -limits, as illustrated above.

We now describe the same region as type 1 region and use the appropriate process of finding the limits.



Now that we are finding the y limits, we just rearrange $x = y/2$ to get $y = 2x$, and the lower limit of y becomes $y = 0$. The limits for x easily follow. So,

$$\begin{aligned} \int_0^2 \int_{y/2}^1 e^{x^2} dx dy &= \iint_R e^{x^2} dA = \int_0^1 \int_0^{2x} e^{x^2} dy dx \\ &= \int_0^1 \left[e^{x^2} y \right]_{y=0}^{2x} dx = \int_0^1 2xe^{x^2} dx \\ &= e^{x^2} \Big|_0^1 = e - 1 \end{aligned}$$

Notice how easily it is to integrate our function $z = e^{x^2}$ with respect to y first. That is the whole point of reversing the order of integration.