Multiple Integrals

Simple Examples of Double Integrals

We get a first run through of evaluating double integrals by two simple examples, one from a simple expression and another from a finding volume problem.

Example 1:
We are to evaluate the double integral

$$\iint_R y^2 x dA$$

over the rectangle $R = \left\{ \left(x, y\right) : -3 \le x \le 2, 0 \le y \le 1 \right\}$.

Here is our standard double integral expression. First and foremost, notice that unlike single variable calculus, there are two 'components' to any double integral problem, the function *f*(*x,y*) we are integrating, in this case $y^2 x$, and the region *R*. It doesn't make much sense if we are to find the double integral of a function, yet without a region *R* specified making it unclear as to what region to integrate over.

Using our previous theorem, the value of the double integral may be obtained by either iterated integrals

$$\int_{-3}^{2}\int_{0}^{1} y^2 x dy dx \text{ or } \int_{0}^{1}\int_{-3}^{2} y^2 x dx dy$$

bearing in mind that for a rectangular region (which is in this case), the order of the repeated integrals does not matter.

Using the former iterated integrals, we have

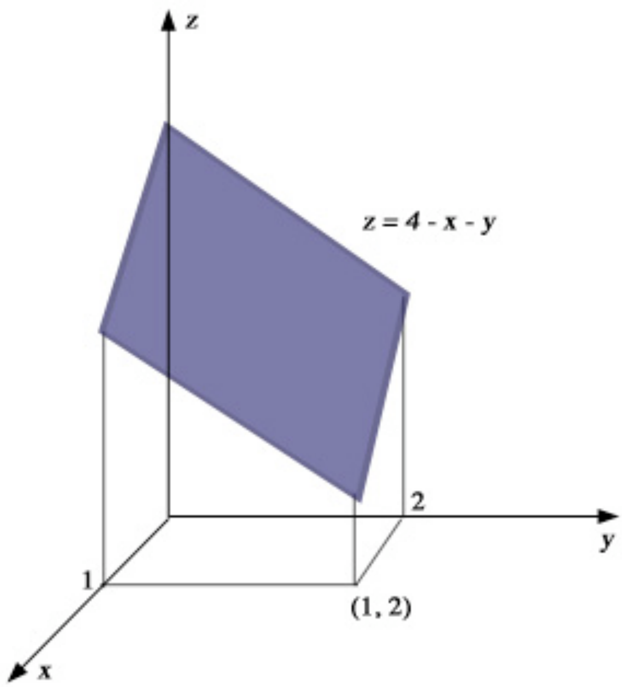
$$\iint_R y^2 x dA = \int_{-3}^{2}\int_{0}^{1} y^2 x dy dx = \int_{-3}^{2}\left[\frac{1}{3} y^3 x\right]_{y=0}^{1} dx$$

$$= \int_{-3}^{2} \frac{1}{3} x dx = \left[\frac{x^2}{6}\right]_{-3}^{2} = -\frac{5}{6}$$

A point to know is that after the first integration, we are substituting the limits into y as shown by the $y = 0$. Since there are two variables in the square bracket, this is to make it unambiguous as to which variable we are substituting the limits into.

With a quick check with the later iterated integral, one would reach the same answer.

Example 2:
We shall now use a double integral to find the volume of the solid that is bounded above by the plane $z = 4 - x - y$ and below by the rectangle $R = \{(x, y) : 0 \le x \le 1, 0 \le y \le 2\}$.



The above graph gives us a feel of the geometrical meaning of the double integral. The function $z = 4 - x - y$ represents the plane which bounds the solid from above and region R is the rectangle which bounds the solid from below. One may say that region R is the *projection* of the plane onto the x-y plane.

$$\text{Vol} = \iint_R (4 - x - y)dA = \int_0^2 \int_0^1 (4 - x - y)dxdy$$

$$= \int_0^2 \left[4x - \frac{x^2}{2} - xy \right]_{x=0}^1 dy = \int_0^2 \left(\frac{7}{2} - y \right) dy$$

$$= \left[\frac{7}{2}y - \frac{y^2}{2} \right]_0^2 = 5$$

Again, we would get the exact same result if we integrated with respect to y first and then with respect to x.