

**AMC**

Polynomials and their Zeros 2000 AMC10 #24

Suppose that  $P(x/3) = x^2 + x + 1$ . What is the sum of all values of  $x$  for which  $P(3x) = 7$ ?

Solution

The idea here is that we want to express the polynomial as  $P(x)$  in terms of  $x$ . The way to do it is called the 'switching of variable'. A full example of this method is shown in solving the AMC Functions question 1991 ASHME#21.

By inspection and some clever rearranging, we can still get the intended result. Note that

$$P\left(\frac{x}{3}\right) = x^2 + x + 1 = 9\left(\frac{x}{3}\right)^2 + 3\left(\frac{x}{3}\right) + 1$$

so

$$P(x) = 9x^2 + 3x + 1 \quad \text{and}$$

$$P(3x) = 81x^2 + 9x + 1$$

If  $7 = P(3x) = 81x^2 + 9x + 1$ , then solving for  $x$  gives

$$0 = 81x^2 + 9x - 6 = 81\left(x^2 + \frac{1}{9}x - \frac{2}{27}\right)$$

The zero-coefficient relationship for quadratic polynomials implies that the sum of the zeros of the quadratic is the negative of the linear term that is within the parentheses. Hence the sum of all the values of  $x$  for which  $P(3x) = 7$  is  $-1/9$ .