

AMC

Polynomials and their Zeros 2001 AMC12 #19

The polynomial $P(x) = x^3 + ax^2 + bx + c$ has the property that the average of its zeros, the product of its zeros, and the sum of its coefficients are all equal. The y -intercept of the graph of $y = P(x)$ is 2. What is b ?

Solution

In order to find b , it seems obvious that we may need to find the other unknowns a and c . The strategy here is to use the information we have to find these unknowns in a systematic manner. It is also advise that the student be familiar with the zero-coefficient relationships which are so important in all types of polynomials questions.

Basically, our information of this polynomials is

1. The average of its zeros, the product of its zeros, and the sum of its coefficients are all equal.
2. The y -intercept is 2.

After some thinking we can easily get c from information number 2. By putting $x = 0$, which is where the y -intercept occurs, we have $c = P(0) = 2$. Hence $c = 2$.

Our polynomials is now $P(x) = x^3 + ax^2 + bx + 2$. Mathematical, point number 1 is

$$\frac{r_1 + r_2 + r_3}{3} = r_1 r_2 r_3 = 1 + a + b + c$$

Here, we have to employ the zero-coefficient relationships to advance further. It allows us to express the terms $r_1 r_2 r_3$ in a , b , and c , which leads to the solution. Recall that for a polynomial of the 3rd degree

$$P(x) = x^3 + ax^2 + bx + c$$

$$r_1 + r_2 + r_3 = -a$$

$$r_1r_2 + r_1r_3 + r_2r_3 = b$$

$$r_1r_2r_3 = -c$$

where $r_1r_2r_3$ are the first, second and third roots and taking note of the negative sign first. Substituting this into the previous equation,

$$\frac{-a}{3} = -c = a + b + c$$

which makes it a breeze to solve.

$$a = -3(-c) = -3(-2) = 6$$

We just use the other equation and solve for b .

$$1 + a + b + c = -c$$

$$b = -2c - 1 - a = -4 - 1 - 6 = -11$$

The answer is -11 . I advise that the student be familiar with the zero-coefficient relationship for the general polynomial.

For a polynomial of the form

$$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

The coefficients and related to the zeros by

$$-a_{n-1} = r_1 + r_2 + \cdots + r_n = (\text{the sum of all the zeros})$$

$$a_{n-1} = r_1r_2 + r_1r_3 + \cdots + r_{n-1}r_n$$

$$= (\text{the sum of the zeros taken two at a time})$$

or in general

$$(-1)^i a_{n-i} = r_1r_2r_3 \cdots r_i + r_1r_2r_4 \cdots r_{n-i+1} + \cdots + r_{n-i+1} \cdots r_{n-2}r_{n-1}r_n$$

$$= (\text{the sum of the zeros taken } i \text{ at a time})$$